ESTIMATES OF SOME APPLICABLE INEQUALITIES ON TIME SCALES

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Abstract. The main objective of the paper is to establish explicit estimates on some applicable inequalities in two variables on time scales which can be used in the study of certain qualitative properties of dynamical equations on time scales.

1. Introduction

Many physical, chemical and biological phenomena can be modeled using dynamic equations and study of such problems has enormous potential. In 1988 Stefan Hilger [10] in his Ph.D thesis introduced the calculus on time scales which unifies the continuous and discrete analysis. As a response to the diverse need of the applications recently in last decade many authors have studied the properties of solutions of dynamic equations on time scales [1, 2, 3, 4, 7, 8, 9, 11, 12, 13, 14, 15, 16, 17, 18]. Motivated by the above results in this paper we find inequalities with explicit estimates which can found to be important tool in the study of dynamical systems on time scales. Let $\mathbb{R}$ denotes the set of real numbers and $\mathbb{T}$ denotes an arbitrary time scale.

More basic information about time scales calculus can be found in monographs [5, 6]. Now following [17, 18] we give some basic definitions about calculus on time scales in two variables.

We say that $f : \mathbb{T} \to \mathbb{R}$ is rd-continuous provided f is continuous at each right-dense point of $\mathbb{T}$ and has a finite left sided limit at each left dense point of $\mathbb{T}$. C_{rd} denotes the set of rd-continuous function defined on $\mathbb{T}$. Let $\mathbb{T}_1$ and $\mathbb{T}_2$ be two time scales with at least two points and consider the time scales intervals $\overline{\mathbb{T}}_1 = [x_0, \infty) \cap \mathbb{T}_1$ and $\overline{\mathbb{T}}_2 = [y_0, \infty) \cap \mathbb{T}_2$ for $x_0 \in \mathbb{T}_1$ and $y_0 \in \mathbb{T}_2$ and $\Omega = \mathbb{T}_1 \times \mathbb{T}_2$. Let $\sigma_1, \rho_1, \Delta_1$ and $\sigma_2, \rho_2, \Delta_2$ denote the forward jump operators, backward jump operators and the delta differentiation operator respectively on $\mathbb{T}_1$ and $\mathbb{T}_2$. Let $a < b$ be points in $\mathbb{T}_1$, $c < d$ are point in $\mathbb{T}_2$, $[a, b)$ is the

2010 *Mathematics Subject Classification.* 26E70, 34N05.

Key words and phrases. integral equations, explicit estimate, integral inequality, continuous dependence, time scale.

half closed bounded interval in $\mathbb{T}_1$, and $[c, d)$ is the half closed bounded interval in $\mathbb{T}_2$.

We say that a real valued function f on $\mathbb{T}_1 \times \mathbb{T}_2$ at $(t_1, t_2) \in \overline{\mathbb{T}}_1 \times \overline{\mathbb{T}}_2$ has a Δ_1 partial derivative $f^{\Delta_1}(t_1, t_2)$ with respect to t_1 if for each $\epsilon > 0$ there exists a neighborhood U_{t_1} of t_1 such that

$$\left| f\left(\sigma_1\left(t_1\right), t_2\right) - f\left(s, t_2\right) - f^{\Delta_1}\left(t_1, t_2\right)\left(\sigma_1\left(t_1\right) - s\right)\right| \leq \varepsilon \left|\sigma_1\left(t_1\right) - s\right|,$$

for each $s \in U_{t_1}, t_2 \in \mathbb{T}_2$. We say that f on $\mathbb{T}_1 \times \mathbb{T}_2$ at $(t_1, t_2) \in \overline{\mathbb{T}}_1 \times \overline{\mathbb{T}}_2$ has a Δ_2 partial derivative $f^{\Delta_2}(t_1, t_2)$ with respect to t_2 if for each $\eta > 0$ there exists a neighborhood U_{t_2} of t_2 such that

$$\left| f\left(t_1, \sigma_2\left(t_2\right)\right) - f\left(t_1, l\right) - f^{\Delta_2}\left(t_1, t_2\right)\left(\sigma_2\left(t_2\right) - l\right)\right| \leq \eta \left|\sigma_2\left(t_2\right) - l\right|,$$

for all $l \in U_{t_2}, t_1 \in \mathbb{T}_1$. The function f is called rd-continuous in t_2 if for every $\alpha_1 \in \mathbb{T}_1$, the function $f(\alpha_1, .)$ is rd-continuous on $\mathbb{T}_2$. The function f is called rd-continuous in t_1 if for every $\alpha_2 \in \mathbb{T}_2$ the function $f(., \alpha_2)$ is rd-continuous on $\mathbb{T}_1$.

The partial delta derivative of $z(x, y)$ for $(x, y) \in \Omega$ with respect to x, y and xy is denoted by $z^{\Delta_1}(x, y)$, $z^{\Delta_2}(x, y)$, $z^{\Delta_1\Delta_2}(x, y) = z^{\Delta_2\Delta_1}(x, y)$.

2. Main Results

Now we give our main results.

Theorem 2.1. Let $u, p \in C_{rd}(\Omega, \mathbb{R}_+)$ and $k \geq 0$ is constant. If

$$u(x, y) \leq k + \int_{x_0}^{x}\int_{s_0}^{s}\int_{y_0}^{y} p(\eta, \tau) u(\eta, \tau)\, \Delta\tau\Delta\eta\Delta s, \tag{2.1}$$

for $(x, y) \in \Omega$, then

$$u(x, y) \leq k e_{\int_{s_0}^{s}\int_{y_0}^{y} p(\eta,\tau)\Delta\tau\Delta\eta.}(x, x_0). \tag{2.2}$$

Proof. Assume $k > 0$. Define a function $w(x, y)$ by right hand side of (2.1), $w(x, 0) = w(0, y) = k$, $u(x, y) \leq w(x, y)$.

$$w^{\Delta_2}(x, y) = \int_{x_0}^{x}\int_{s_0}^{s} p(\eta, \tau) u(\eta, \tau)\, \Delta\eta\Delta\tau, \tag{2.3}$$

$$w^{\Delta_1}(x, y) = \int_{x_0}^{x}\int_{y_0}^{y} p(\eta, \tau) u(\eta, \tau)\, \Delta\tau\Delta\eta, \tag{2.4}$$

$$w^{\Delta_1\Delta_1}(x, y) = \int_{y_0}^{y} p(x, \tau) u(x, \tau)\, \Delta\tau, \tag{2.5}$$

and

$$w^{\Delta_1\Delta_1\Delta_2}(x,y) = p(x,y)\,u(x,y) \leq p(x,y)\,w(x,y)\,. \tag{2.6}$$

From (2.6) and from the facts that $w^{\Delta_1\Delta_1}w(x,y) \geq 0, w^{\Delta_1}w(x,y) \geq 0, w(x,y) > 0$ we have

$$\frac{w^{\Delta_1\Delta_1\Delta_2}(x,y)}{w(x,y)} \leq p(x,y) + \left[\frac{w^{\Delta_1\Delta_1}(x,y)\,w^{\Delta_2}(x,y)}{w^2(x,y)}\right]$$

$$\frac{w^{\Delta_1\Delta_1\Delta_2}(x,y)}{w(x,y)} \leq p(x,y)\,. \tag{2.7}$$

By keeping x fixed we set $y=\tau$ and then delta integrating with respect to τ from y_0 to y and $w^{\Delta_1\Delta_1}(x,y_0)=0$ we get

$$\frac{w^{\Delta_1\Delta_1}(x,y)}{w(x,y)} \leq \int_{y_0}^{y} p(x,\tau)\,\Delta\tau. \tag{2.8}$$

From (2.8) and as we have $w^{\Delta_1}(x,y) \geq 0$, $w(x,y)>0$ we get

$$\frac{\partial}{\Delta_1}\left(\frac{w^{\Delta_1}(x,y)}{w(x,y)}\right) \leq \int_{y_0}^{y} p(x,\tau)\,\Delta\tau. \tag{2.9}$$

By taking y fixed in (2.9) set $x=\eta$ integrating η with respect to x_0 to x and $z^{\Delta_1}(y_0,y)=0$ we have

$$\frac{w^{\Delta_1}(x,y)}{w(x,y)} \leq \int_{x_0}^{x}\int_{y_0}^{y} p(\eta,\tau)\,\Delta\tau\Delta\eta. \tag{2.10}$$

From (2.10) we get

$$w(x,y) \leq ke_{\int_{s_0}^{s}\int_{y_0}^{y} p(\eta,\tau)\Delta\tau\Delta\eta}(x,x_0)\,. \tag{2.11}$$

Using (2.11) in $u(x,y) \leq w(x,y)$ we get the result.

Theorem 2.2. Let p,q be positive and rd-continuous and q be non decreasing. If

$$u(x,y) \leq q(x,y) + \int_{x_0}^{x}\int_{s_0}^{s}\int_{y_0}^{y} p(\eta,\tau)\,u(\eta,\tau)\,\Delta\tau\Delta\eta\Delta s, \tag{2.12}$$

for $(x,y)\in\Omega$, then

$$u(x,y) \leq q(x,y)\,e_{\int_{s_0}^{s}\int_{y_0}^{y} p(\eta,\tau)u(\eta,\tau)\Delta\tau\Delta\eta}(x,x_0)\,. \tag{2.13}$$

Proof. Let $q(x,y) \geq 0$ for $(x,y) \in \Omega$. Then from (2.12), it is easy to see that

$$\frac{u(x,y)}{q(x,y)} \leq 1 + \int_{x_0}^{x} \int_{s_0}^{s} \int_{y_0}^{y} p(\eta,\tau) \frac{u(\eta,\tau)}{q(\eta,\tau)} \Delta\tau\Delta\eta\Delta s. \tag{2.14}$$

Now an application of inequality in Theorem 2.1 gives the result (2.13).

Theorem 2.3. Let $u, g, h, p \in C_{rd}(\Omega, \mathbb{R}_+)$ and $L \in C_{rd}(\Omega \times \mathbb{R}_+, \mathbb{R}_+)$

$$0 \leq L(x,y,u) - L(x,y,v) \leq H(x,y,v)(u-v), \tag{2.15}$$

and $u \geq v \geq 0$, where $H \in C_{rd}(\Omega \times \mathbb{R}_+, \mathbb{R}_+)$. If

$$u(x,y) \leq g(x,y) + h(x,y) \int_{x_0}^{x} \int_{s_0}^{s} \int_{y_0}^{y} L(\eta,\tau,u(\eta,\tau)) \Delta\tau\Delta\eta\Delta s, \tag{2.16}$$

for $(x,y) \in \Omega$ then

$$u(x,y) \leq g(x,y) + h(x,y) \left(\int_{x_0}^{x} \int_{s_0}^{s} \int_{y_0}^{y} L(\eta,\tau,g(\eta,\tau)) \Delta\tau\Delta\eta\Delta s \right)$$
$$\times e_{\int_{s_0}^{s} \int_{y_0}^{y} H(\eta,\tau,g(\eta,\tau))h(\eta,\tau)\Delta\tau\Delta\eta}(x,x_0), \tag{2.17}$$

for $(x,y) \in \Omega$.

Proof. Define a function $w(x,y)$ by

$$w(x,y) = \int_{x_0}^{x} \int_{s_0}^{s} \int_{y_0}^{y} L(\eta,\tau,u(\eta,\tau)) \Delta\tau\Delta\eta\Delta s, \tag{2.18}$$

then $w(x,y_0) = w(x_0,y) = 0$ and inequality (2.16) becomes

$$\begin{aligned} w(x,y) \leq & \int_{x_0}^{x} \int_{s_0}^{s} \int_{y_0}^{y} \{L(\eta,\tau,g(\eta,\tau) + h(\eta,\tau) w(\eta,\tau)) \\ & - L(\eta,\tau,g(\eta,\tau)) + L(\eta,\tau,g(\eta,\tau))\} \Delta\tau\Delta\eta\Delta s \\ \leq & \int_{x_0}^{x} \int_{s_0}^{s} \int_{y_0}^{y} H(\eta,\tau,g(\eta,\tau)) h(\eta,\tau) w(\eta,\tau) \Delta\tau\Delta\eta\Delta s \\ & + \int_{x_0}^{x} \int_{s_0}^{s} \int_{y_0}^{y} L(\eta,\tau,g(\eta,\tau)) \Delta\tau\Delta\eta\Delta s. \end{aligned} \tag{2.19}$$

It can be easily seen that the first term on the right hand side of (2.19) is nonnegative and non decreasing. Now suitable application of Theorem 2.2 to (2.19) we get (2.17).

Theorem 2.4. Let u, g, h, p be as in theorem 2.3. If

$$u(x,y) \le g(x,y) + h(x,y) \int_{x_0}^{x} \int_{s_0}^{s} \int_{y_0}^{y} p(\eta,\tau) u(\eta,\tau) \Delta\tau\Delta\eta\Delta s, \quad (2.19)$$

for $(x,y) \in \Omega$, then

$$u(x,y) \le g(x,y) + h(x,y) \left(\int_{x_0}^{x} \int_{s_0}^{s} \int_{y_0}^{y} p(\eta,\tau) g(\eta,\tau) \Delta\tau\Delta\eta\Delta s \right) \times e_{\int_{s_0}^{s}\int_{y_0}^{y} p(\eta,\tau)h(\eta,\tau)\Delta\tau\Delta\eta}(x,x_0), \quad (2.20)$$

for $(x,y) \in \Omega$.

Proof. Now putting $L(\eta,\tau,u(\eta,\tau)) = p(\eta,\tau) u(\eta,\tau)$ in above Theorem 2.3, we get the result.

3. Applications

Consider integral equation on time scales of the form

$$u(x,y) = g(x,y) + \int_{x_0}^{x} \int_{s_0}^{s} \int_{y_0}^{y} K(x,y,\eta,\tau,u(\eta,\tau)) \Delta\tau\Delta\eta\Delta s, \quad (3.1)$$

where u is unknown function to be found for given $g \in C_{rd}(\Omega,\mathbb{R})$ and $K \in C_{rd}(\Omega\times\Omega\times\mathbb{R},\mathbb{R})$.

Now we give the estimates on the solutions of equation (3.1).

Theorem 3.1. Let g, K in (3.1) satisfy the condition

$$|K(x,y,\eta,\tau,u) - K(x,y,\eta,\tau,v)| \le q(x,y)\, r(\eta,\tau)\, |u-v|, \quad (3.2)$$

$$\left| g(x,y) + \int_{x_0}^{x} \int_{s_0}^{s} \int_{y_0}^{y} K(x,y,\eta,\tau,0) \Delta\tau\Delta\eta\Delta s \right| \le p(x,y), \quad (3.3)$$

where $p, q, r \in C_{rd}(\Omega,\mathbb{R}_+)$. If $u(x,y)$ is solution of (3.1) for $(x,y) \in \Omega$, then

$$|u(x,y)| \le p(x,y) + q(x,y) \left(\int_{x_0}^{x} \int_{s_0}^{s} \int_{y_0}^{y} r(\eta,\tau) p(\eta,\tau) \Delta\tau\Delta\eta\Delta s \right) \times e_{\int_{s_0}^{s}\int_{y_0}^{y} r(\eta,\tau)q(\eta,\tau)\Delta\tau\Delta\eta}(x,x_0), \quad (3.4)$$

for $(x, y) \in \Omega$.

Proof. We have $u(x, y)$ as solution of (3.1) for $(x, y) \in \Omega$. We have

$$u(x,y) \le \left| g(x,y) + \int_{x_0}^{x}\int_{s_0}^{s}\int_{y_0}^{y} K(x,y,\eta,\tau,0)\Delta\tau\Delta\eta\Delta s \right|$$

$$+ \int_{x_0}^{x}\int_{s_0}^{s}\int_{y_0}^{y} |K(x,y,\eta,\tau,u(\eta,\tau)) - K(x,y,\eta,\tau,0)|\Delta\tau\Delta\eta\Delta s$$

$$\le p(x,t) + q(x,t)\int_{x_0}^{x}\int_{s_0}^{s}\int_{y_0}^{y} r(\eta,\tau)|u(\eta,\tau)|\Delta\tau\Delta\eta\Delta s. \tag{3.5}$$

Now applying Theorem 2.4 to (3.5) gives (3.4).

A function $u \in C_{rd}(\Omega, \mathbb{R})$ is called ϵ approximate solution of equation (3.1) if their exists a $\epsilon \ge 0$ such that

$$\left| u(x,y) - \left\{ g(x,y) + \int_{x_0}^{x}\int_{s_0}^{s}\int_{y_0}^{y} K(x,y,\eta,\tau,u(\eta,\tau))\Delta\tau\Delta\eta\Delta s \right\} \right| \le \epsilon, \tag{3.6}$$

for $(x, y) \in \Omega$.

Now we estimate the difference between two approximate solution of (3.1).

Theorem 3.2. Let $u_i(x, y)(i = 1, 2)$ be ϵ_i approximate solutions of (3.1) for $(x, y) \in \Omega$. Suppose function K satisfies the condition (3.2). Then

$$|u_1(x,y) - u_2(x,y)| \le (\varepsilon_1 + \varepsilon_2)[1 + q(x,y)\left(\int_{x_0}^{x}\int_{s_0}^{s}\int_{y_0}^{y} r(\eta,\tau)w(\eta,\tau)\Delta\tau\Delta\eta\Delta s \right)$$

$$\times e_{\int_{s_0}^{s}\int_{y_0}^{y} r(\eta,\tau)q(\eta,\tau)\Delta\tau\Delta\eta}(x, x_0), \tag{3.7}$$

for $(x, y) \in \Omega$.

Proof. Since $u_i(x, y)(i = 1, 2)$ be ϵ_i approximate solutions of (3.1) we get

$$\left| u_i(x,y) - \left\{ g(x,y) + \int_{x_0}^{x}\int_{s_0}^{s}\int_{y_0}^{y} K(x,y,\eta,\tau,u_i(\eta,\tau))\Delta\tau\Delta\eta\Delta s \right\} \right| \le \varepsilon_i. \tag{3.8}$$

From (3.8) and using the inequalities

$$|v-\overline{v}| \leq |v| + |\overline{v}|\,, \quad |v| - |\overline{v}| \; \leq |v-\overline{v}|\,, \tag{3.9}$$

we have

$$\begin{aligned}
\epsilon_1+\epsilon_2 \geq & \left| u_1(x,y) - \left\{ g(x,y) + \int_{x_0}^{x}\int_{s_0}^{s}\int_{y_0}^{y} K(x,y,\eta,\tau,u_1(\eta,\tau))\,\Delta\tau\Delta\eta\Delta s \right\} \right| \\
& + \left| u_2(x,y) - \left\{ g(x,y) + \int_{x_0}^{x}\int_{s_0}^{s}\int_{y_0}^{y} K(x,y,\eta,\tau,u_2(\eta,\tau))\,\Delta\tau\Delta\eta\Delta s \right\} \right| \\
& \geq \left| \left[u_1(x,y) - \left\{ g(x,y) + \int_{x_0}^{x}\int_{s_0}^{s}\int_{y_0}^{y} K(x,y,\eta,\tau,u_1(\eta,\tau))\,\Delta\tau\Delta\eta\Delta s \right\} \right] \right. \\
& - \left. \left[u_2(x,y) - \left\{ g(x,y) + \int_{x_0}^{x}\int_{s_0}^{s}\int_{y_0}^{y} K(x,y,\eta,\tau,u_2(\eta,\tau))\,\Delta\tau\Delta\eta\Delta s \right\} \right] \right| \\
& \geq |u_1(x,y) - u_2(x,y)| - \left| \int_{x_0}^{x}\int_{s_0}^{s}\int_{y_0}^{y} K(x,y,\eta,\tau,u_1(\eta,\tau))\,\Delta\tau\Delta\eta\Delta s \right. \\
& - \left. \int_{x_0}^{x}\int_{s_0}^{s}\int_{y_0}^{y} K(x,y,\eta,\tau,u_2(\eta,\tau))\,\Delta\tau\Delta\eta\Delta s \right|.
\end{aligned} \tag{3.10}$$

Let $w(x,y) = |u_1(x,y) - u_2(x,y)|$ for any $(x,y) \in \Omega$. From (3.10) and using (3.2), we have

$$\begin{aligned}
w(x,y) \leq & (\epsilon_1+\epsilon_2) \\
& + \int_{x_0}^{x}\int_{s_0}^{s}\int_{y_0}^{y} |K(x,y,\eta,\tau,u_1(\eta,\tau)) - K(x,y,\eta,\tau,u_2(\eta,\tau))| \Delta\tau\Delta\eta\Delta s \\
& \leq (\varepsilon_1+\varepsilon_2) + q(x,y) \int_{x_0}^{x}\int_{s_0}^{s}\int_{y_0}^{y} r(\eta,\tau)\, w(\eta,\tau)\,\Delta\tau\Delta\eta\Delta s.
\end{aligned} \tag{3.11}$$

Now an using inequality in Theorem 2.3 yields the result.

Remark. In case $u_1(x, y)$ is a solution of (3.1) then we have $\epsilon_1 = 0$ and from (3.7) we have $u_2(x, y) \to u_1(x, y)$ as $\epsilon_2 \to 0$. If we put $\epsilon_1 = \epsilon_2 = 0$ in (3.7) then the uniqueness of solution of equation (3.1) is established.

ACKNOWLEDGEMENTS. The research in the present paper is supported by Science and Engineering Research Board(SERB, New Delhi, India), File No. SR/S4/MS-861/13.

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